Here $\xi = r/R$, $\xi_0 = r_\sigma/R$, $KR = \tau_0/\rho gR$ is the surface-curvature parameter. The upper sign in expressions (12), (13), and (15) relates to the inside-face problem, the lower sign to the outside-face problem.

The results of a numerical integration are presented in Figs. 2 and 3. The behavior of the velocity W_0 of the quasisolid core relative to the wall and the zone boundary ξ_0 is qualitatively similar to the plane case. The other parameters being equal, the maximum values W_0^{max} and $|1 - \xi_0^{max}|$ for a film on the inside surface of a cylinder are higher and displaced to the right as compared with the case of flow off the outside surface. In this connection the mass of liquid shaken off is greater for the film flowing off the inside surface of the cylinder (Fig. 4). As the deceleration parameter A increases, so does the mass of liquid shaken off. The rate of increase is particularly significant up to $A \approx 10$.

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VARIATIONAL SOLUTION OF EQUATION OF NONLINEAR MASS AND ENERGY TRANSFER

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The use of a variational principle of Hamilton type is considered for problems of nonlinear mass transfer in a semibounded plate with constant and variable diffusional properties.

In complex cases associated with heat and mass transfer in systems with chemical transformations or polymerization processes, in biological systems, and in special cases of catalytic processes, significant deviation from the Fourier and Fick laws is observed. Processes of heat and mass transfer of this kind may be mathematically described by an equation of the form

$$\frac{\partial \varphi}{\partial \tau} + \operatorname{div}(\vec{v}\varphi) + \tau^* \frac{\partial^2 \varphi}{\partial \tau^2} + \operatorname{div}[k(\varphi)(\operatorname{grad}\varphi)^n] = F(\varphi), \quad (1)$$

$$n \leqslant 1.$$

This equation may be obtained on the assumption that the flux of material is determined by an expression of the form [1]

$$\vec{j} = -k(\varphi)(\operatorname{grad}\varphi)^n - \tau^* \dot{\vec{j}}.$$

In the case of heat transfer, φ represents the thermal energy; in the case of mass transfer, the concentration. For the heat-transfer equation $F(\varphi)$ is a heat source or sink and for the mass-transfer equation a mass source or sink due to chemical transformations.

Consider the case of mass and energy transfer in a semibounded plate with variable diffusional properties in the presence of a chemical reaction; in this case, Eq. (1) takes the form

$$\frac{\partial C}{\partial \tau} = \frac{\partial}{\partial x}\left[D^*(C)\left(\frac{\partial C}{\partial x}\right)^n\right] + kC^m, \quad n \leqslant 1, \quad m = 1, 2, 3, \quad (2)$$

where k is the rate constant of the chemical reaction; m is the order of the chemical reaction. In the general case it is expedient to assume that the order of the reaction may be either integral or fractional.

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The concentration distribution is assumed to be uniform at the initial moment; i.e., at $\tau = 0$, $C(x, 0) = 0$.

In accordance with the idea of the penetration depth of the concentration front, the parameter $\delta(\tau)$, the extension of the region with nonzero concentration, is introduced. For $x > \delta(\tau)$ the plate has a uniform initial mass concentration and there is no mass transfer.

A boundary condition of the first kind specifies the concentration distribution over the surface of the plate at any moment of time; in this semibounded plate the concentration distribution at the boundary is

$$C(0, \tau) = \theta(\tau); \; C(\delta(\tau), \tau) = 0; \; 0 \leqslant x \leqslant \delta(\tau). \tag{3}$$

A variational principle of Hamilton type [3-7] will be used to solve Eq. (2) with the boundary conditions in Eq. (3). This method is characterized by the introduction in the Lagrangian of a parameter λ, which tends to zero when the variational process is complete. By this means it is possible to obtain an artificial approximation to problems for which there are no classical integrals of Hamilton type.

The approximating solution of Eqs. (2) and (3) is sought in the form

$$C(x, \tau) = \theta(\tau)\mu^p; \; \mu = 1 - \frac{x}{\delta(\tau)}; \tag{4}$$

p is a positive integer.

Consider the Lagrangian

$$L = \left[\frac{D^*(C)}{2(n+1)}\left(\frac{\partial C}{\partial x}\right)^{n+1} - \frac{\lambda D^*(C)}{2}\left(\frac{\partial C}{\partial \tau}\right)^2 - \frac{kD^*(C)}{m+1}C^{m+1}\right]\exp\left(\frac{\tau}{\lambda}\right). \tag{5}$$

The corresponding Euler–Lagrange equation is

$$\lambda\frac{\partial^2 C}{\partial \tau^2} + \frac{\partial C}{\partial \tau} = \frac{\partial}{\partial x}\left[D^*(C)\left(\frac{\partial C}{\partial x}\right)^n\right] + kC^m. \tag{6}$$

When $\lambda \to 0$, Eq. (6) reverts to Eq. (2). Solving Eqs. (6) and (3) is equivalent to finding the minimum of the functional

$$I = \int_{\tau_0}^{\tau_1}\int_0^{\delta(\tau)} L\,dx\,d\tau, \tag{7}$$

where the time integral is chosen arbitrarily.

Let

$$D^*(C) = k_0\sum_{i=0}^{\infty} a_i\mu^{pi} = k_0[a_0 + a_1\mu^p + a_2\mu^{2p} + \ldots], \tag{8}$$

where k_0 and a_i are constants.

After evaluating the inner integral, Eq. (7) takes the form

$$I^* = \int_{\tau_0}^{\tau_1}\left[-\frac{Ak_0^2 p^{n+1}\theta^{n+1}(\tau)}{2(n+1)\delta^n(\tau)} + \frac{Bk_0\lambda\theta'^2(\tau)\delta(\tau)}{2} + \frac{Dk_0\lambda p^2\theta^2(\tau)\delta'^2(\tau)}{\delta(\tau)} + \right.$$
$$\left. + Ek_0\lambda p\theta(\tau)\theta'(\tau)\delta'(\tau) + \frac{Fk_0 k\theta^{m+1}(\tau)\delta(\tau)}{m+1}\right]\exp\left(\frac{\tau}{\lambda}\right)d\tau, \tag{9}$$

where

$$A = \left(\sum_{i=0}^{\infty}\frac{a_i}{m(n+2i+1)-n}\right)^2;$$

$$B = \sum_{i=0}^{\infty}\frac{a_i}{p(2+i)+1};$$

$$D=\sum_{i=0}^{\infty}\frac{a_i}{[p(i+2)-1]\,p(i+2)\,[p(i+2)+1]};$$

$$E=\sum_{i=0}^{\infty}\frac{a_i}{(2+i)\,p\,[p(2+i)+1]};$$

$$F=\sum_{i=0}^{\infty}\frac{a_i}{p(m+i+1)+1}.$$

The necessary condition for an extremum of the functional is that its variation should vanish; i.e., the Euler–Lagrange equation must be satisfied on the curve realizing the extremum of the functional [2].

The appropriate Euler–Lagrange equation for Eq. (9) is

$$\frac{Ak_0np^{n+1}\theta^{n+1}(\tau)}{2(n+1)\delta^{n+1}(\tau)}+\frac{Fkk_0\theta^{m+1}(\tau)}{m+1}-\frac{2Dk_0p^2\theta^2(\tau)\delta'(\tau)}{\delta(\tau)}-$$

$$-Ek_0p\theta(\tau)\theta'(\tau)=\lambda\left\{\left[\frac{2Dk_0p^2\theta^2(\tau)\delta'(\tau)}{\delta(\tau)}+Ek_0p\theta(\tau)\theta'(\tau)\right]'-\right.$$

$$\left.-\frac{Bk_0\theta'^2(\tau)}{2}-\frac{Dk_0p^2\theta^2(\tau)\delta'^2(\tau)}{\delta^2(\tau)}\right\}. \tag{10}$$

Letting λ tend to zero, the result is an ordinary differential equation of the form

$$\frac{d}{d\tau}(\delta^{n+1}(\tau))+\left[\frac{E(n+1)\theta'(\tau)}{2Dp\theta(\tau)}-\frac{Fk(n+1)\theta^{m+1}(\tau)}{2D(m+1)p^2}\right]\delta^{n+1}(\tau)=\frac{Ak_0np^{n-1}\theta^{n-1}(\tau)}{4D}. \tag{11}$$

Solving this equation gives

$$\delta^{n+1}(\tau)=\frac{Ak_0np^{n-1}}{4D}\cdot\exp\int_{\tau_0}^{\tau_1}\left[\frac{Fk(n+1)\theta^{m-1}(\tau)}{2D(m+1)p^2}-\frac{E(n+1)\theta'(\tau)}{2Dp\theta(\tau)}\right]d\tau\times$$

$$\times\int_{\tau_0}^{\tau_1}\exp\left\{\int_{\tau_0}^{\tau_1}\left[\frac{E(n+1)\theta'(\tau)}{2Dp\theta(\tau)}-\frac{Fk(n+1)\theta^{m-1}(\tau)}{2D(m+1)p^2}\right]d\tau\right\}\theta^{n-1}(\tau)\,d\tau, \tag{12}$$

and $\delta(\tau_0)=0$.

Using Eqs. (12) and (4), an approximating solution for Eqs. (2) and (3) is obtained.

To estimate the error of this solution, Eq. (2) is transformed using the substitution

$$\lambda_1=\frac{x}{\delta(\tau)},\quad \bar{\tau}=\tau. \tag{13}$$

Successive evaluation gives

$$\frac{\partial C}{\partial\bar{\tau}}=\theta'(\tau)(1-\lambda_1)^p+\theta(\tau)\,p(1-\lambda_1)^{p-1}\lambda_1\frac{\delta'(\tau)}{\delta(\tau)}, \tag{14}$$

$$\frac{\partial C}{\partial\lambda_1}=-\frac{p\theta(\tau)(1-\lambda_1)^{p-1}}{\delta(\tau)}, \tag{15}$$

$$\frac{\partial^2 C}{\partial\lambda_1^2}=\frac{p(p-1)\,\theta(\tau)(1-\lambda_1)^{p-2}}{\delta^2(\tau)} \tag{16}$$

and it will now be expedient to introduce the notation

$$\varepsilon=\frac{\partial C}{\partial\bar{\tau}}-\frac{\partial}{\partial\lambda_1}\left[D^*(C)\left(\frac{\partial C}{\partial\lambda_1}\right)^n\right]-kC^m. \tag{17}$$

Substituting Eqs. (14)-(16) into Eq. (17) gives the mean-square value of the error

$$J=\frac{\theta'^2(\tau)}{2p+1}+\frac{\theta(\tau)\theta'(\tau)\delta'(\tau)}{(2p+1)\delta(\tau)}+\frac{p\theta^2(\tau)\theta'^2(\tau)}{(4p^2-1)\delta^2(\tau)}+\frac{k^2}{2pm+1}-$$

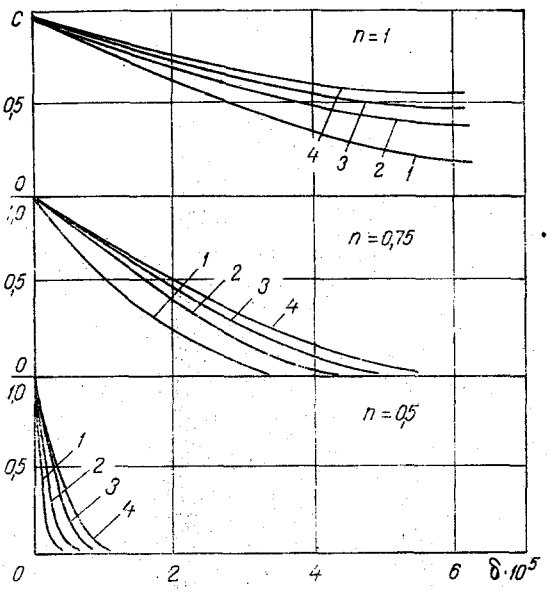

Fig. 1. Concentration distribution for τ = 1 (1), 2 (2), 3 (3), and 4 sec (4); p = 3; $D^* = 10^{-9}$ m^2/sec.

$$-\frac{2k}{p+pm+1}\left[\theta'(\tau)+\frac{\theta(\tau)\theta'(\tau)}{(m+1)\delta(\tau)}\right]+$$

$$+\frac{k_0^2 p^{2n}\theta^{2n}(\tau)}{\delta^{2n}(\tau)}\left\{(Q_1+Q_2)\left(1+\frac{n^2(p-1)^2}{\delta^2(\tau)}\right)-\frac{a_0 a_1}{(p-1)(1+2n)}+\right.$$

$$\left.+\frac{n(p-1)}{\delta(\tau)}\left[\frac{a_0^2 n(p-1)}{(2pn-2n-1)\delta(\tau)}-Q_2\right]\right\}+$$

$$+\frac{2k_0 p^{n+1}\theta^{n+1}(\tau)\delta'(\tau)}{\delta^{n+1}(\tau)}\left[Q_4\left(1-\frac{n(p-1)}{\delta(\tau)}\right)-\frac{a_0 n}{(n-1)(pn-n+p)\delta(\tau)}\right]+$$

$$+\frac{2k_0 p^n\theta^n(\tau)}{\delta^n(\tau)}\left[Q_3\theta'(\tau)\left(1-\frac{n(p-1)}{\delta(\tau)}\right)-\frac{n(p-1)}{\delta(\tau)}\left(\frac{a_0\theta'(\tau)}{pn-n+p}+kQ_5\right)\right]. \qquad (18)$$

where

$$Q_1=\sum_{i=1}^{\infty}\frac{a_i^2}{2i-1+2pn-2n};$$

$$Q_2=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{a_i a_j}{(i+j)p-1+2pn-2n};$$

$$Q_3=\sum_{i=1}^{\infty}\frac{a_i}{ip+pn-n+p};$$

$$Q_4=\sum_{i=1}^{\infty}\frac{a_i}{(ip-1+pn-n+p)(ip+pn-n+p)};$$

$$Q_5=\sum_{i=0}^{\infty}\frac{a_i}{pn-n+ip+pm}.$$

Consider the particular case of Eq. (1) corresponding to the nonlinear mass transfer in a semibounded plate with constant diffusional properties; in this case, Eq. (1) takes the form

TABLE 1. Values of the Error J

$n=1$		$n=0.75$		$n=0.5$	
p	$J\tau^2$	p	$J\tau^2$	p	$J\tau^2$
1	0,0633	1	0,1088	1	0,1481
2	0,0067	2	0,0237	2	0,0363
3	0,0058	3	0,0051	3	0,0139

$$\frac{\partial C}{\partial \tau} = -\frac{\partial}{\partial x}\left[D^*\left(-\frac{\partial C}{\partial x}\right)^n\right], \quad n \leqslant 1. \tag{19}$$

It is possible to solve Eqs. (19) and (3) using a variational principle of Hamilton type, the approximating solution being written in the form in Eq. (4). The Lagrangian in Eq. (5) takes the form

$$L_1 = \left[\frac{D^*}{n+1}\left(-\frac{\partial C}{\partial x}\right)^{n+1} - \frac{\lambda}{2}\left(\frac{\partial C}{\partial \tau}\right)^2\right]\exp\left(\frac{\tau}{\lambda}\right). \tag{20}$$

Then, following the known scheme [3], $\delta(\tau)$ may be evaluated,

$$\delta(\tau) = \left[\frac{D^* p^n n (4p^2-1)}{pn-n+p}\left(\int_{\tau_0}^{\tau} 0^{\frac{4pn-n-1}{2p}}(\tau)\,d\tau\right)0^{\frac{(2p-1)(n+1)}{2p}}(\tau)\right]^{\frac{1}{n+1}}, \tag{21}$$

and $\delta(\tau_0) = 0$.

Using Eqs. (21) and (4), an approximating solution of Eqs. (19) and (3) is obtained.

The mean-square error may be evaluated in accordance with Eq. (18), which in the present case reduces to the form

$$J = \frac{0'^2(\tau)}{2p+1} + \frac{D^{*2}n^2p^{2n}\theta^{2n}(\tau)(p-1)^2}{(2pn-2n-1)\delta^{2n+2}(\tau)} + \frac{0'(\tau)0(\tau)\delta'(\tau)}{(2p+1)\delta(\tau)} +$$
$$+ \frac{p\theta^2(\tau)\delta'^2(\tau)}{(4p^2-1)\delta^2(\tau)} - \frac{2D^*np^n(p-1)0'(\tau)0^n(\tau)}{(pn-n+p)\delta^{n+1}(\tau)} - \frac{2p^{n+1}D^*n\theta^{n+1}(\tau)\delta'(\tau)}{(n+1)(pn-n+p)\delta^{n+2}(\tau)}.$$

As an example, Fig. 1 shows the solution of Eqs. (19) and (3) for $\Theta(\tau) = 1$ with different degrees of nonlinearity n.

One of the main problems in the practical use of the variational method is the choice of the order of the approximating polynomial. In [3] much attention was given to the question of the optimum profile. It was shown that the optimum profile is a polynomial of first or second order for the case of the energy distribution. Solutions of the linear equation obtained by different methods were compared in [9, 10]. In [4-7] parabolic profiles of the energy or concentration distribution were chosen for the solution of various linear problems.

Despite these results, approximating polynomials of higher order were used in [3, 8].

In [8] a cubic profile of the energy distribution was chosen in a variational solution using a local potential. In [3] the nonlinear heat conduction in a semibounded body was calculated by a variational method of Hamilton type using a cubic polynomial for the profile of the energy distribution. The approximate solution obtained is in good agreement with the accurate solution.

To solve Eqs. (19) and (3) a cubic profile of the concentration distribution was chosen. Table 1 gives values of the error for various n and p. An analysis of these errors leads to the choice p = 3 for the solution of the problem.

NOTATION

C, concentration; $\theta(\tau)$, concentration at the left-hand boundary of the region considered; D*, diffusion coefficient; $k(\varphi)$, transfer coefficient; $\vec{j}$, flux of material; $\vec{v}$, velocity; p, order of approximating polynomial; x, coordinate; τ, time; τ^*, relaxation time; τ_0, τ_1, boundary values of time interval; $\delta(\tau)$, penetration depth of concentration front; λ, Lagrangian multiplier.

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HEAT TRANSFER IN TURBULENT FLOW OF POLYATOMIC GASES ALONG A TUBE

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A numerical method is used to calculate the transfer to ammonia for different models of turbulent viscosity. The results obtained are compared with experimental data.

With the increase in heat-flux levels in various power stations and the growing variety of heat carriers (polyatomic and chemically reacting gases, material in a near-critical and supercritical state, etc.), there has arisen a need for methods of calculation of the turbulent flow along a tube of liquids with strongly variable physical properties.

Both physical and mathematical difficulties beset the solution of this problem. Because there is no consistent theory of turbulence at present, it remains uncertain whether semiempirical models of turbulence developed for flows of incompressible liquid along a tube may be used for liquids with variable properties. From a mathematical viewpoint, the existing temperature and pressure dependences of the physical properties of the liquid lead to "strong" nonlinearity of the initial system of equations, so that it is necessary to use finite-difference methods for its solution.

Such methods have been used to obtain solutions for turbulent flows of gaseous nitrogen and air [1] and hydrogen in a state of equilibrium dissociation [2] along a circular tube.

In [1], a comparative analysis of 11 different models of turbulent viscosity was made on the basis of experimental data. It was shown that in the conditions under consideration the formula of [4] gives the best agreement with experiment [3].

In [2] turbulent viscosity was determined using the Reichardt formula [5] with Goldman's correction [6].

In both cases it was assumed that the turbulent analog of the Prandtl number is unity, that the gas is perfect, and that the pressure dependence of the thermodynamic properties and molecular transfer coefficients is negligible. Note that the last two assumptions considerably restrict the use of these methods, and the conclusions drawn as to their applicability require further verification.

The present paper outlines a finite-difference method that may be used to calculate turbulent flows in circular tubes for arbitrary temperature and pressure dependences of the thermodynamic and transfer properties of the gas.

The results obtained for the heat transfer to ammonia using the formulas of [4, 5] and two variants of the Millionshchikov formula [7, 6] to determine the turbulent viscosity are compared with experimental data [8]. The Millionshchikov formula is of great practical

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